On the flow in channels when rigid obstacles are placed in the stream

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SUMMARY

A previous paper drew attention to the collective importance of three physical quantities Q, R, S associated with ideal fluid flow in a horizontal channel. Invariability of these quantities at different cross-sections of the flow implies respectively conservation of flow rate, energy and momentum; and their values determine a wave-train uniquely. The properties of Q, R, S are recalled in the present paper to account for the various effects of lowering a rigid obstacle into a stream. The conditions giving rise to dissimilar types of flow are examined; in particular, the circumstances causing stationary waves on the downstream side are clearly distinguished from those under which the receding stream assumes a uniform 'supercritical' state. A well-known result in the theory of the solitary wave is shown to apply to the receding stream even when the extreme conditions for the wave are exceeded; although it fails to account for a region close to the obstacle where the curvature of the streamlines becomes large. In passing, a feature of the theory is shown to bear on the practical problem of producing a uniform stream. Precise calculations are made for the flow under a vertical sluice-gate and under an inclined plane. To account for the region near the bottom edge of the sluice-gate, a method based on conformal transformation is used whereby an unknown curve in the hodograph plane is approximated by an arc of an ellipse. The accuracy of the results is more than sufficient for practical purposes, and they compare favourably with solutions previously obtained by relaxation methods. A number of experiments with water streams are described.

1. INTRODUCTION

When the stream in an open horizontal channel is spanned by a fixed cylindrical obstacle, the effects of the obstruction may extend to great distances both upstream and downstream. Furthermore, each part of the stream may take several distinct forms depending on the cross-sections of the cylinder, the extent of its immersion, and the nature of the undisturbed stream. Kelvin (1886) demonstrated that a train of periodic waves may form downstream, whose amplitude increases with the drag on the obstacle (i.e. with the 'wave resistance'). His theoretical treatment was limited to cases where the waves are small in both amplitude and length; but Benjamin & Lighthill (1954) have shown in general that the wave resistance may take any value up to a certain maximum. If the obstacle is lowered sufficiently far into the stream, energy losses may occur due to breaking of waves, and further immersion may lead to the formation of a fully turbulent 'bore or' 'hydraulic jump' behind which waves appear to be absent. The action of a fixed obstacle can also bring about a transition from a uniform subcritical stream (of which the speed is less than $\surd(g \times \text{depth})$) to a supercritical one. This effect is best known through the example of a vertical sluice-gate, which is discussed in many text-books on hydraulics. The conditions under which waves can form ahead or in the rear of a sluice-gate have not, up to the present time, been satisfactorily explained.

The first object of this paper is to present a unified account of these various effects, and to make clear the relation amongst them. A general method for calculating the form of the receding stream when waves are absent will also be outlined, and will be illustrated by applications to the flow under a sluice-gate and under a planing surface. The treatment is based on ideal-fluid theory, but a number of experiments have been made on water streams in order to test the theoretical results; these experiments will be described in the concluding part of the paper. A fact emerging incidentally from the discussion appears to have an important bearing on the practical problem of producing a uniform horizontal stream of water. Such streams are often desired for tests on ship models, and for this reason they were made the subject of numerous experiments by Binnie, Davies & Orkney (1955). These authors found it possible to produce a satisfactory subcritical stream without measurable waves, provided the Froude number ($F = \text{speed}/\surd(g \times \text{depth})$) was less than about 0·5. At Froude numbers exceeding this value waves always appeared, despite a variety of measures aimed at avoiding them. In agreement with this result, the present theory indicates that flows converging towards a uniform stream are impossible at Froude numbers somewhat less than unity. This fact might be relevant to the design of a channel for model tests over a range of speeds.

In the course of examining the asymptotic properties of a disturbed stream, we shall show that a formula known in the theory of the solitary wave also applies to flows beyond the extreme conditions for the wave. It is subsequently used to calculate the form of the converging stream which issues from under a sluice-gate. The flow in the immediate neighbourhood of the gate has to be treated in another way; but when suitably combined, the two methods account satisfactorily for the whole stream. The results of the calculations demonstrate the variation of the contraction ratio with Froude number, and hence establish the relation between the sluice-opening and the total discharge. They are apparently more accurate than previous estimates, notably those due to Pajer (1937), and the single result which Southwell & Vaisey (1946) calculated by relaxation methods. The latter authors drew attention to the difficulties which confront their methods when applied to flow near points of separation from solid boundaries (p. 160 in their paper). These difficulties appear to be responsible for most of the discrepancy (about 1·5%) between their computed value of the sluice-opening and the value according to the present theory. From a practical point of view there is admittedly little justification for carrying the calculations to high orders of approximation, since the theory does not apply with precision to real fluids. In the experiments described later, for instance, the thickness of the boundary layer on the channel bottom was observed to be about one-fifteenth of the sluice-opening, thus indicating that frictional effects were far more significant than the final error in the ideal-fluid calculations. An accurate form of the theory is nevertheless desirable, for at least it gives assurance that the discrepancies observed in real flows are due to friction alone.

The problem to be considered is indicated in figure 1. We shall assume that the motion is steady, and that both far upstream and far downstream

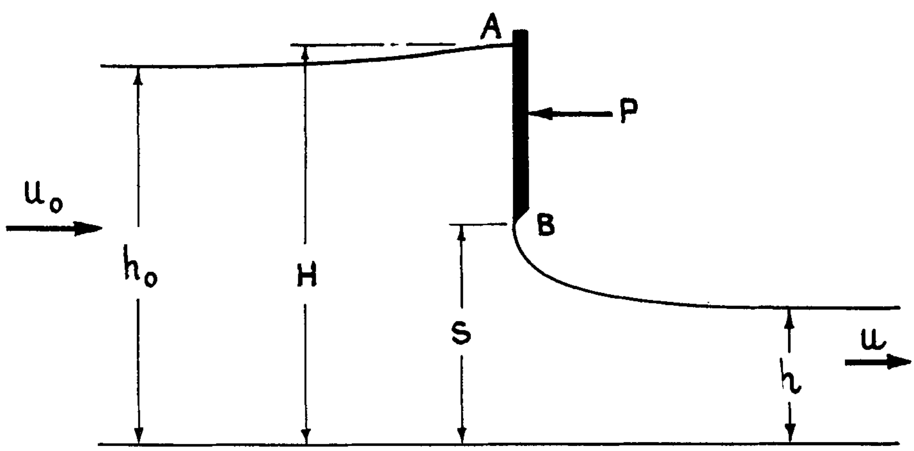

Figure 1. Side elevation of sluice-gate.

the flow is uniform and horizontal, having depth and velocity h_0, u_0 and h, u respectively. In the paper already cited, Southwell & Vaisey noted the existence of a symmetrical solution in which the free surface regains its original level after passing the sluice; but flow of this kind does not occur in practice unless there is some severe obstruction downstream. The free surface usually falls to a lower level, and Binnie (1952) has shown that the approaching stream must then be subcritical ($F < 1$) and the receding stream supercritical. The condition of continuity is

$$u_0 h_0 = uh = Q, \tag{1.1}$$

where Q is the discharge per unit span; and Bernoulli's equation may be written

$$gh_0 + \tfrac{1}{2}u_0^2 = gh + \tfrac{1}{2}u^2 = gH, \tag{1.2}$$

where H is the total head measured above the bottom of the channel. If both Q and H are given, h_0, h, u_0 and u can be calculated from (1.1) and (1.2), but there is no simple way of finding the sluice-opening s. Alternatively, if s is specified together with either Q or H, as is usual in a practical example, elementary methods fail to give any of the other quantities. In addition to (1.1) and (1.2) a third simple relation may be written down equating the reduction in flux of momentum to the resultant of horizontal forces, which include the force P on unit span of the gate and the pressure forces in the oncoming and receding streams; thus, denoting density by ρ, we have

$$\rho(u_0^2 h_0 - u^2 h) = P - \tfrac{1}{2}\rho g(h_0^2 - h^2). \tag{1.3}$$

However, since P is unknown, this equation is of no assistance in the main problem.

At point A in figure 1 the free surface rises to stagnation level, and the velocity there is zero. At point B it must slope vertically, since otherwise the bounding streamline would make a sharp corner there and the velocity would become infinite, which is clearly impossible if the surface is free. One may nevertheless expect that the free surface has infinite curvature at B, for this property has been established in many similar instances when gravity is absent (see, for example, Southwell & Vaisey, p. 160).

2. General theory of the flow

The investigation is conveniently begun by considering the different types of flow which may exist at points far removed from the obstacle. In other words, we shall first examine the possible asymptotic forms of the flow upstream and downstream. At the outset it is necessary to emphasize that the stream cannot be *perfectly* uniform at any finite distance from the obstacle. This fact becomes obvious when one considers that the hodograph variable $\zeta = qe^{-i\theta}$ (where q and θ are the magnitude and direction of the velocity) is analytic within the stream; therefore, by virtue of an elementary property of analytic functions, ζ cannot be constant throughout any finite region unless it is constant everywhere. Having thus dismissed the case of a uniform stream, there remain only two types of flow capable of extension to indefinitely large distances upstream or downstream. The first possibility is a train of periodic waves, and the second a steadily converging stream resembling the outskirts of a solitary wave. In practice, flow of the latter sort may be indistinguishable from a uniform stream except in a region close to the obstacle; strictly speaking, however, they become uniform only at infinity. Although this distinction is a trivial one in some respects, it will assume importance in the subsequent argument.

It is now desirable to recall some of the ideas put forward by Benjamin & Lighthill (1954) in connection with the theory of gravity waves of finite amplitude. They pointed out that for any two-dimensional steady flow in a horizontal channel, there are three important physical quantities which in the absence of friction and horizontal external forces, have the same value at every cross-section of the flow. These are Q the volume flow per unit span, R the energy per unit mass (i.e. g times the total head as measured above the channel bottom), and S the resultant of momentum flux and pressure force per unit span divided by the density. A train of long waves was shown to be determined uniquely by the values of Q, R and S, and there is reason for believing that this property is common to all wave-trains in parts distant from their origin and termination. This view is supported by the recent work of De (1955), who calculated numerical values of the quantities in question for a wide range of wavelengths. The former authors also demonstrated that the physically realizable combinations of Q, R and S are confined within certain limits. For instance, if Q and R are fixed, then S is restricted between a higher value corresponding to uniform subcritical flow and a lower value corresponding to uniform supercritical flow. Stationary wave-trains may occur for intermediate values of S; but the wave amplitude tends to zero as the upper limit is approached, whereas at the lower limit the wavelength becomes infinite, and the only possible wave is the solitary wave.

We proceed by extending a line of argument begun in the paper by Benjamin & Lighthill (p. 455). If a rigid obstacle is lowered gradually into a *slightly* subcritical stream, it experiences an increasing wave resistance (corresponding to a reduction of S on the downstream side) until the waves formed downstream become of great length, and their profile approaches that of the solitary wave. A different situation arises, however, if the Froude number of the respective supercritical flow (i.e. the flow with the same values of Q and R as the subcritical flow upstream) is greater than the value 1·25, at which, as McCowan (1894) showed, the solitary wave takes its extreme sharp-crested form. A gradual approach to the lower value of S is now impossible, since the extreme condition for the solitary wave is exceeded at the limit. In fact, as S is reduced from its upper limit, periodic 'waves of maximum height' occur at a value of S in excess of the lower limit, and a further gradual reduction will result in energy losses due to breaking of waves. The obstacle must then be lowered by a finite amount, bringing S to the minimum value, before steady lossless flow is again possible. No wave can form on the supercritical stream under these circumstances; but it will be shown presently that the stream has some features in common with the solitary wave. Recognition of the lower limit to S also shows that any obstruction of a stream initially in a supercritical condition is bound to result in energy losses, since a reduction of S is otherwise impossible. In fact a bore will be formed upstream. A bore sometimes occurs in practice on the stream receding from a sluice-gate. It may be caused by some obstruction, or simply by friction if the channel is fairly long. The flow assumes a subcritical state behind the bore, but with a smaller value of R than the subcritical flow upstream from the gate. (The material of this paragraph may be made somewhat clearer by consulting figure 1 in the paper by De (1955).)

The condition $F > 1{\cdot}25$ noted above may be shown, by means of (1.1) and (1.2), to correspond to the condition $F_0 < 0{\cdot}792$ for the respective subcritical stream. Hence we can conclude that if $F_0 > 0{\cdot}792$ a transition from subcritical to supercritical uniform flow, in the manner indicated by figure 1 (present paper), cannot be achieved with R unchanged, since only wave-trains can result on the downstream side. Although such a transition is possible if $F_0 < 0{\cdot}792$, the obstacle must be immersed to a considerably greater extent than that which causes waves. Thus a clear distinction is made between the action of a sluice-gate and that of a slightly submerged obstacle causing a wave-train downstream. These considerations seem to supply a complete answer to the question whether waves can form on the downstream side of a sluice-gate. This matter was raised by Southwell & Vaisey (1946), who were led to expect waves in the course of their calculations, although none were found. It was also considered by Binnie (1952), who presented a tentative argument indicating that waves are impossible. The question whether waves can form on the upstream side is deferred to a later part of the discussion.

The special case of critical flow ($F = 1$) deserves separate attention. Let us therefore consider the flow in a channel which is supplied from a large reservoir, so that R is fixed, and which is initially free from obstruction of any sort. Such circumstances are well-known to give rise to critical flow, which makes the value of Q the maximum possible at the given total head. This fact may be established by various methods, a review of which was given by Binnie (1949). Suppose now that an obstacle is fixed in slight contact with the free surface at a point well down the channel, thereby causing a small reduction in S on the downstream side. The disturbed flow is clearly unstable; for the change in S necessitates a decrease in Q from its maximum value, since R cannot change, and hence an increase in depth on the upstream side. As the free surface rises upstream, a greater force is exerted against the obstacle, and the downstream value of S further decreases. Before steady conditions can be resumed, the free surface must rise to stagnation level on the front of the obstacle. The elevation of stagnation level above the channel bottom (i.e. the total head H) is easily shown to be $1{\cdot}5$ times the depth of critical flow. Thus the obstacle is 'wetted' to a height $\frac{1}{2}H$ in the final steady state. The resistance experienced by an obstacle this far immersed is much too large to be ascribable to wave formation; and the flow is therefore of the kind illustrated in figure 1.

It is now proposed to examine the profile of the receding stream in the case where waves are absent. The work of Benjamin & Lighthill again provides a convenient starting point. In a new presentation of 'cnoidal wave' theory, they demonstrated that the condition of constant Q and S leads to an approximate differential equation for the free surface (the condition of constant R being superfluous). In terms of coordinates (x, y) with the x-axis along the channel bottom, this may be written

$$\tfrac{1}{3}Q^2\left(\frac{dy}{dx}\right)^2 + gy^3 - 2Ry^2 + 2Sy - Q^2 = 0. \tag{2.1}$$

The nature of the approximation on which (2.1) is based is noteworthy. The assumption was made that successive derivatives of the slope y' diminish fairly rapidly in order of magnitude, so that, for instance, y^2y^{iv} can reasonably be neglected in comparison with y''. The next stage of approximation was observed to contribute terms of the order of y'^4 to (2.1), and further stages would evidently contribute terms of the order of successively higher powers of y'^2. The trend of the coefficients in the early stages suggests that the method of approximation is rapidly convergent, provided y' is fairly small. There is accordingly some justification for assuming that an exact solution exists for any flow with constant Q, R and S, and that (2.1) will provide a close approximation to it in parts where y' is small. As will be explained in the next paragraph, this equation can lead to an approximation for the solitary wave. The mathematical existence of this wave has been proved by Friedrichs & Hyers (1954), and so the validity of (2.1) is confirmed in this one respect. Its validity in the present application, however, cannot be established by appeal to the work of Friedrichs & Hyers; for the solitary wave is impossible with the values of Q, R and S to be considered here. Nevertheless, it may be said that the physical aspects of the present problem, in which the flow-pattern is supposed to be stationary relative to the obstacle, are a good deal clearer than these aspects of the solitary wave, which is stationary only with respect to a hypothetical frame of reference.

If Q, R and S take the values

$$Q = uh, \qquad R = gh+\tfrac{1}{2}u^2, \qquad S = \tfrac{1}{2}gh^2+u^2h, \tag{2.2}$$

which are appropriate to a uniform stream with depth h and velocity u, and whose Froude number is therefore $F = u/\sqrt{(gh)}$, the cubic expression in (2.1) factorizes and the whole equation can be arranged in the form

$$\left(\frac{dy}{dx}\right)^2 = \frac{3}{F^2}\left(\frac{y}{h}-1\right)^2\left(F^2-\frac{y}{h}\right). \tag{2.3}$$

With $F > 1$ this is identical with the equation for the solitary wave obtained by Rayleigh (1876a). Other than the uniform flow $y = h$, its only solution is

$$\frac{y}{h} = 1+(F^2-1)\,\mathrm{sech}^2\left[\frac{\sqrt{(3F^2-3)}x}{2Fh}\right]. \tag{2.4}$$

The solitary wave is thus the only wave which can emerge from a uniform supercritical stream without loss of energy or momentum.

This approximation to the solitary wave is most accurate at the outskirts of the wave where the slope and curvature are small. When the wave approaches its extreme form (at $F = 1{\cdot}25$), the approximation is found to be no longer accurate over the whole wave, yet still holds at the outskirts. It is clear also that these results apply to the outlying parts of a stream with constant Q, R and S which converges towards uniform flow at $F > 1{\cdot}25$, although the solitary wave is then impossible. The streams of concern here are clearly included in this category. Accordingly, their free surfaces must satisfy (2.4) approximately up to a region close to the obstacle where the slope or curvature becomes large. There is a simple way of testing this assertion; for, as Addison (1938) and Binnie (1952) have pointed out, the flow under a sluice-gate when the ratio s/H is small resembles a jet issuing under pressure in the absence of gravity. The latter example is illustrated in figure 2 (a), where the bottom line may be taken to represent either a plane wall or the centre-line of a symmetrical jet. Rayleigh (1876 b) found the complete solution of this problem. The free surface satisfies the equation

$$\frac{y}{d} = 1 - \frac{2}{\pi}\sin\theta, \tag{2.5}$$

where θ is the angular slope $\tan^{-1}(dy/dx)$ which ranges between 0 and $-\frac{1}{2}\pi$, and d has the same meaning as h in figure 1. In the corresponding gravity problem F is very large, and (2.3) consequently gives

$$\frac{y}{h} = 1 - \frac{1}{\sqrt{3}}\tan\theta. \tag{2.6}$$

Agreement between the two results is partly attributable to the similarity of the numerical factors $1/\sqrt{3}$ and $2/\pi$ appearing in (2.5) and (2.6), but is in fact better than this alone would suggest. The Cartesian equations equivalent to (2.5) and (2.6) are easily found; hence the agreement can be checked very readily by plotting the two curves on tracing paper and fitting them together. They are found to fit closely in parts where the magnitude of θ is less than about 30°, the maximum discrepancy in y/h being about 1%. The curves diverge significantly in the higher region where the curvature of the first of them (from (2.5)) becomes large. Note that for smaller Froude numbers the accuracy of (2.4) is likely to improve, whereas (2.5) becomes inaccurate.

The approximation of (2.4) may be tested in one further instance by comparison with the solution for a sluice-gate which Southwell & Vaisey (1946) obtained by relaxation methods. In their example the assumed conditions were, in the notation of figure 1, $H = 12$ and $h_0 = 11$ length units. The depth and Froude number of the flow far downstream were not given explicitly, but from (1.1) and (1.2) they are easily calculated to be $h = 3{\cdot}854$ length units and $F = 2{\cdot}055$. When these values are put into (2.4), the curve obtained from this equation is found to coincide with the lower part of the profile given by Southwell & Vaisey (p. 151 in their paper), but begins to diverge perceptibly from it at a point where the slope is about $-30°$, a little way below the highest point.

We turn from the main argument for a moment in order to consider another matter explained by the differential equation (2.3). If $F < 1$ the equation has no real solution except $y = h$, the case of uniform flow. This fact demonstrates convincingly that non-uniform flow converging towards a uniform stream, as discussed previously, is impossible for values of F less than unity yet large enough for a long-wave approximation to apply. Any attempt to produce a uniform stream under these conditions is therefore bound to fail, unless the flow can be made uniform on entry into the channel. Clearly, the result of any non-uniformity on entry will be a train of periodic waves, the values of Q, R and S being differently correlated than for uniform flow. The occurrence of waves as predicted here was observed experimentally by Binnie, Davies & Orkney (1955), who mentioned similar findings by other investigators.

This is a suitable place to raise the question of wave formation on the upstream side of an immersed obstacle. The situation at small Froude numbers, to which a short wavelength approximation can be applied, was discussed at length by Lamb (1932, §§ 242–5), whose treatment was restricted, however, to infinitesimal disturbances of the free surface. He noted that in the absence of dissipative forces the problem is to some extent indeterminate, since waves of small amplitude can always be superimposed on the stream without altering certain of the physical conditions (e.g. Q or S). There appears to be no simple way of extending Lamb's argument to account for waves of finite size; but the presence or absence of waves seems likely to depend only on the conditions specified far upstream. Hence one is justified in assuming that the flow is uniform upstream, as is done in the following treatment of the sluice-gate problem.

3. Flow under a sluice-gate

Equation (2.4) has been shown to apply behind, but not too near, any obstacle adequately immersed. Other methods are needed, however, to deal with the rapidly curving portion of the free surface which is always to be expected near a 'trailing edge', and which clearly depends on the shape of the obstacle. A sluice-gate and planing surface are the only examples treated in this paper, but similar calculations could undoubtedly be carried out for other boundary shapes.

It is reasonable to suppose that in the region of B in figure 1 the flow is little affected by gravity, since the acceleration of the fluid particles is large there. Although (2.5) will roughly give this part of the free surface, the solution of the problem indicated in figure 2 (b) is rather more accurate. Here the flow upstream from the jet is confined within parallel walls at a distance D from the centre-line; and, as a basis for comparison with the example in figure 1, we may take $D = H$ as a useful estimate. This problem was solved by Mises (1917), although a formula for the free surface was not given in his paper. Straightforward methods show, however, that the free surface satisfies the equation

$$y = d - \frac{2d}{\pi m} \tan^{-1}(m \sin \theta), \tag{3.1}$$

with $$m = \frac{2Dd}{D^2 - d^2}, \quad \text{and} \quad \frac{s}{d} = 1 + \frac{2\tan^{-1} m}{\pi m}. \tag{3.2}$$

This result can be combined with (2.3) to obtain a rough estimate of the complete free surface below the sluice. The two curves may be joined at a point where $\theta = -30°$, say, so that in (2.3) $y'^2 = (\tan 30°)^2 = \frac{1}{3}$. To calculate the sluice-opening s for a given value of F, which specifies an appropriate value of D and also the height of the 'joining point' (y_0, say), the two equations (3.2) are used to eliminate k and d from the relation

$$s - y_0 = \frac{2d}{\pi m}(\tan^{-1} m - \tan^{-1} \tfrac{1}{2}m), \tag{3.3}$$

which follows from (3.1). In the example treated by Southwell & Vaisey this method leads to a value 0·602 for the contraction ratio h/s, whereas they

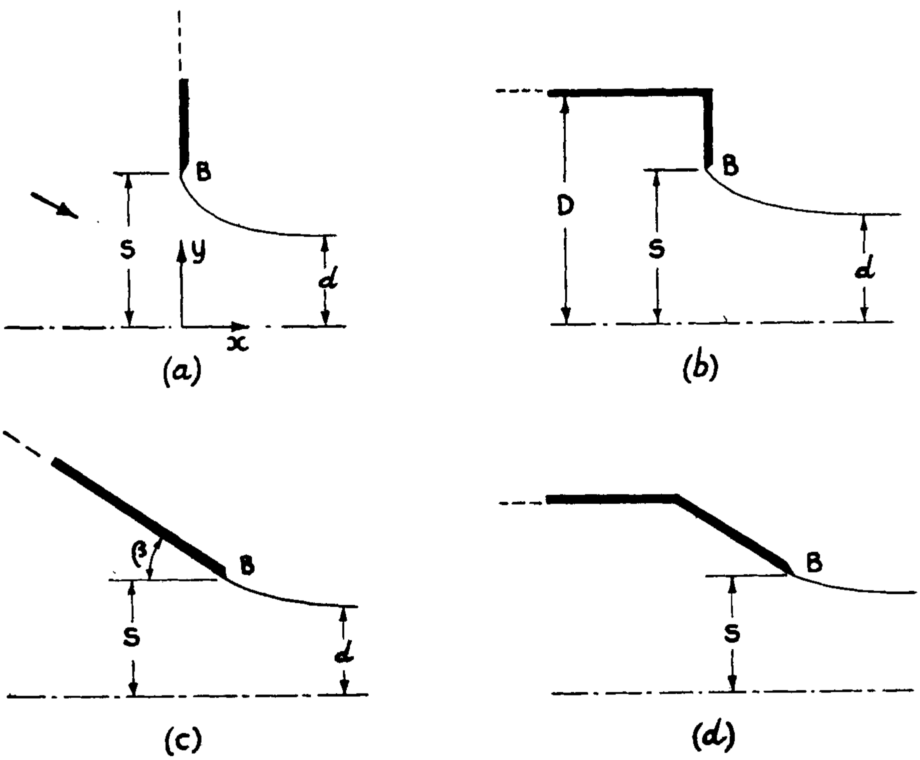

Figure 2. Free streamline problems not affected by gravity.

obtained 0·608.* The method takes no account of velocity changes over the upper portion of the free surface, although over all parts the resultant velocity q varies according to Bernoulli's equation

$$q^2 = 2g(H-y). \tag{3.4}$$

As the free surface falls between heights s and y_0, the velocity increases by a factor $\sqrt{(H-s)}/\sqrt{(H-y_0)}$, which is found to be significantly greater

* Their result was misprinted as 0·66; but Miss Vaisey has confirmed that the intented value is 0·608.

than unity in typical examples. We are therefore led to seek a more accurate method taking full account of gravity.

By the method now to be described, a solution is found which satisfies the boundary condition (3.4) exactly at the edge B and at a 'joining point' below which (2.3) can be applied accurately. The method is developed from one given by Pajer (1937), and so the present account is made fairly brief. Following the usual procedure in dealing with irrotational motions in the plane of a complex variable z, use is made of the fact that since the velocity potential ϕ and stream function ψ are harmonic, the variable $w = \phi + i\psi$ is a function of z only. The first aim of the method is to establish an approximate relation between w and the hodograph variable $\zeta = dw/dz = qe^{-i\theta}$. We consider a transformation to the plane of an auxilliary variable $f(z)$ defined by

$$\frac{p}{\zeta} = \frac{pe^{i\theta}}{q} = f - \frac{k}{f}, \tag{3.5}$$

where p and k are positive real constants. The separation of the real and imaginary parts of (3.5) leads to

$$\frac{p\cos\theta}{q} = \left(|f| - \frac{k}{|f|}\right)\cos\omega, \tag{3.6}$$

$$\frac{p\sin\theta}{q} = \left(|f| + \frac{k}{|f|}\right)\sin\omega, \tag{3.7}$$

where $\omega = \arg(f)$. We also make use of the transformation

$$f' = f^2 + 1/f^2. \tag{3.8}$$

It is now assumed that $\omega = \theta = -\frac{1}{2}\pi$ at the edge B, and that $|f| = 1$ along the free surface below this point. According to (3.5), the circle so defined in the f-plane is mapped in the hodograph as an ellipse, whose size and eccentricity depends on p and k. By a suitable choice of these constants, the points in the hodograph representing B and the 'joining point' may be linked by an arc of the ellipse.

From (3.8) we have that $f' = 2\cos 2\omega$ on the free surface below B. At B, $f' = -2$, since $\omega = -\frac{1}{2}\pi$; and at infinity downstream $f' = 2$, since $\omega = \theta = 0$. In other words, the free surface is mapped in the f'-plane between -2 and 2 along the real axis. On the straight line AB in figure 1, $\omega = \theta = -\frac{1}{2}\pi$; and $q = 0$ at A. Thus AB is mapped between infinity and -2 along the negative real axis of f'. If the free surface upstream from A is taken to be horizontal, we have $\omega = \theta = 0$, and so f' is real along this surface also. It is therefore mapped along the positive real axis between infinity and a value $f'_0 = f_0^2 + 1/f_0^2$ representing the conditions far upstream. It is a simple step to show that the lower half of the f'-plane maps the entire flow-pattern.

The streamline $\psi = 0$ extends along the bed of the channel, and the streamline $\psi = Q$ (where Q is the volume flow rate) forms the upper boundary of the flow. As shown above, the latter streamline is mapped along the real axis in the f'-plane. The figure in the w-plane is an infinite strip of width Q lying on the real (ϕ) axis, and is mapped in the lower half of the f'-plane by the Schwarz–Christoffel transformation

$$w = \phi + i\psi = \frac{Q}{\pi}\log\left(\frac{f'-f_0'}{2-f'}\right). \tag{3.9}$$

The relation between ζ and w is determined by (3.5), (3.8) and (3.9). The form of the free surface below B can hence be deduced by a standard method (see Lamb 1932, § 74), details of which may be omitted here. We find for this surface

$$x = \frac{2cs}{\pi}\left[\operatorname{logtan}\left(\frac{\omega}{2}+\frac{\pi}{2}\right) - n\tanh^{-1}\left(\frac{\cos\omega}{n}\right)\right], \tag{3.10}$$

$$y = s - \frac{2cs}{m\pi}\frac{1+k}{1-k}[\tan^{-1}m + \tan^{-1}(m\sin\omega)], \tag{3.11}$$

where $n = \frac{1}{2}(f_0 + 1/f_0)$, $m = 2f_0/(f_0^2-1)$, and c is the value of y/s for $\omega = 0$.

If the boundary condition (3.4) is to be satisfied at B, we must have, by reason of (3.7),

$$\frac{p}{\sqrt{(2g[H-s])}} = 1+k.$$

If this condition is also to be satisfied at a point below B where $y = y_0$, $\theta = \theta_0$, and $\omega = \omega_0$, we must also have

$$\frac{p\sin\theta_0}{\sqrt{(2g[H-y_0])}} = (1+k)\sin\omega_0.$$

It follows that

$$\frac{H-y_0}{H-s} = \left(\frac{\sin\theta_0}{\sin\omega_0}\right)^2, \tag{3.12}$$

in which ω_0 is determined (from (3.6) and (3.7)) by

$$\tan\omega_0 = \frac{1-k}{1+k}\tan\theta_0. \tag{3.13}$$

There will be no need to consider further the constant p, which merely fixes the scale of the figure in the f-plane.

It is convenient to introduce a new constant $\alpha = \tan^{-1}m$ (hence $\cot\frac{1}{2}\alpha = f_0$), in terms of which (3.11) gives

$$y_0 = s - \frac{2cs}{\pi}\frac{1+k}{1-k}\cot\alpha\,[\alpha + \tan^{-1}(\tan\alpha\sin\omega_0)]. \tag{3.14}$$

We also find, by putting $\omega = 0$ in (3.11), that

$$\frac{1}{c} = 1 + \frac{2}{\pi}\frac{1+k}{1-k}\alpha\cot\alpha. \tag{3.15}$$

Furthermore, if h_1 is the value of the depth upstream consistent with the assumption of its being constant everywhere, the continuity condition (1.2) leads to

$$\frac{h_1}{cs} = (f_0-k)/(1-k) = (\cot\tfrac{1}{2}\alpha - k\tan\tfrac{1}{2}\alpha)/(1-k),$$

from which, if a substitution for c is made from (3.14), we obtain

$$k\left(\frac{h_1}{s} - \frac{2h_1}{\pi s}\alpha\cot\alpha - \tan\tfrac{1}{2}\alpha\right) = \frac{h_1}{s} + \frac{2h_1}{\pi s}\alpha\cot\alpha - \cot\tfrac{1}{2}\alpha. \tag{3.16}$$

These results represent a flow which closely resembles the true solution up to the cross-section where $y = y_0$; beyond this the methods described previously can be applied. Our main object is to calculate the contraction ratio h/s as a function of the ratio s/H, which defines the flow-pattern uniquely on the basis of Froude number scaling. It is first necessary to obtain a suitable estimate of the quantity h_1 appearing in (3.16). We have so far neglected the small difference between H and h_0 (see figure 1), but now tentatively take account of it by assigning to h_1 a value somewhere between the two. (Note that no continuity condition is violated by doing this, since we are merely seeking a flow particularly resembling the actual flow in the region just below the sluice.) Inspection of the flow-pattern obtained by Southwell & Vaisey (1946, figure 25) suggests that $\frac{1}{3}(H+2h_0)$ is a fair estimate for h_1, although the value of h_1 is not at all critical when s/H is fairly small. An initial estimate of h/s is required to calculate h_0 from the relation

$$H(h_0+h) = h_0^2 + h_0 h + h^2, \tag{3.17}$$

which follows from (1.2) and (1.3). When θ_0 is specified, y_0/s can be found by eliminating c, k, α and ω_0 amongst the five equations (3.12) to (3.16). The calculation has to be done by numerical methods, but the following procedure was found to give fairly quick results. The value of α is first guessed. This task is aided by extrapolation after solutions have been obtained for some values of s/H, and as a help in this respect we have $\alpha = 0$ for $s/H = 0$. Using this estimate of α, k is calculated directly from (3.16); then c is calculated from (3.15), and ω_0 from (3.13). Finally, y_0 is obtained both from (3.12) and from (3.14). This sequence is repeated with another estimate of α, and it is then possible to assess α more accurately by interpolation designed to bring the alternative estimates of y_0 together. Repetitions of the whole process with successively closer approximations to α lead rapidly to an accurate estimate of y_0. The method is helped by the fact that the value of y_0 given by (3.12) varies greatly with α, whereas the value from (3.14) varies very little with α.

θ	x (ft)	y (ft)	q_1 (f/s)	q_2 (f/s)	q_1-q_2 (f/s)
$-90°$	0	1·0	16·048	16·048	0
$-75°$	0·0024	0·9863	16·085	16·075	0·010
$-60°$	0·0197	0·9464	16·186	16·155	0·031
$-45°$	0·0692	0·8835	16·318	16·280	0·038
$-35°$	0·1324	0·8309	16·417	16·383	0·034
$-25°$	0·2368	0·7719	16·499	16·499	0

Table 1

For example, with $s/H = 0{\cdot}2$ and $\theta_0 = -25°$ it is found that $y_0/s = 0{\cdot}7719$. Coordinates of the free surface in the region where $-90° \leqslant \theta \leqslant -25°$ are given in table 1, which also includes values of q given by (3.6) (q_1), and the correct values according to (3.4); s is taken as 1 ft, and g as 32·19 ft/sec^2. The largest error in q is seen to be slightly greater than 0·2%.

After the value of y_0 has been established in this way, the calculation is completed as follows. If (2.3) is applied at the 'joining point', it gives

$$\frac{3}{F^2}\left(\frac{y_0}{h}-1\right)^2\left(F^2-\frac{y_0}{h}\right) = \left(\frac{dy}{dx}\right)^2 = \tan^2\theta_0.$$

From (1.3) we also have

$$\frac{H}{h} = 1+\tfrac{1}{2}F^2.$$

The elimination of F^2 between these equations leads to

$$(y_0-h)^2(2H-2h-y_0)-\tfrac{2}{3}(H-h)h^2\tan^2\theta_0 = 0. \tag{3.18}$$

When divided by s^3, this equation becomes a cubic in h/s whose coefficients are functions of the known quantities H/s, y_0/s and θ_0; this can be solved in a straightforward manner.

Values of h/s calculated with $\theta_0 = -25°$ are tabulated as a function of s/H in table 2, which includes the Froude number according to (3.18). The table also shows the set of values obtained by Pajer (1937), who used

s/H	h/s	h/s (Pajer)	F
0	0·6110	0·6110	∞
0·1	0·6060	—	5·568
0·2	0·6022	0·6046	3·822
0·3	0·5995	0·6036	3·020
0·4	0·5980	0·6043	2·522
0·5	0·5981	0·6066	2·165

Table 2

a method similar to the first part of the present method but extended over the whole free surface. The errors in velocity were then considerably larger than at present, and were in fact largest in a region corresponding to $\theta \doteqdot -25°$, where, by the present method, the error is deliberately reduced to zero. Moreover, no account was taken of the difference between H and h_0. The discrepancies between Pajer's results and the present ones are fairly small, although quite significant. A remarkable feature of table 2 is the smallness of the variation in h/s. This effect is mainly due to gravity; for in its absence (the case illustrated in figure 2 (*b*)) the contraction ratio increases quite rapidly as the flow is narrowed upstream, and is always greater than the extreme value for the case in figure 2 (*a*).

The present method requires slight modification when applied to the problem treated by Southwell & Vaisey, in which H, h and F are given and s remains to be found. First, an estimate of s/H is made, and then h/s is calculated as before. Since h/s varies quite slowly with s/H, this result leads to a much closer estimate and the approximation is then further improved by repetitions of the process. Starting with $h/H = 0{\cdot}3212$ and

$F^2 = 4{\cdot}227$, which are the values appropriate to Southwell & Vaisey's example, the method gives finally $h/s = 0{\cdot}5991$. We recall that the value obtained by them was 0·608. The curves derived from the present theory and by Southwell & Vaisey are both shown in figure 3. The two are indistinguishable up to a region just below B, but diverge appreciably in this

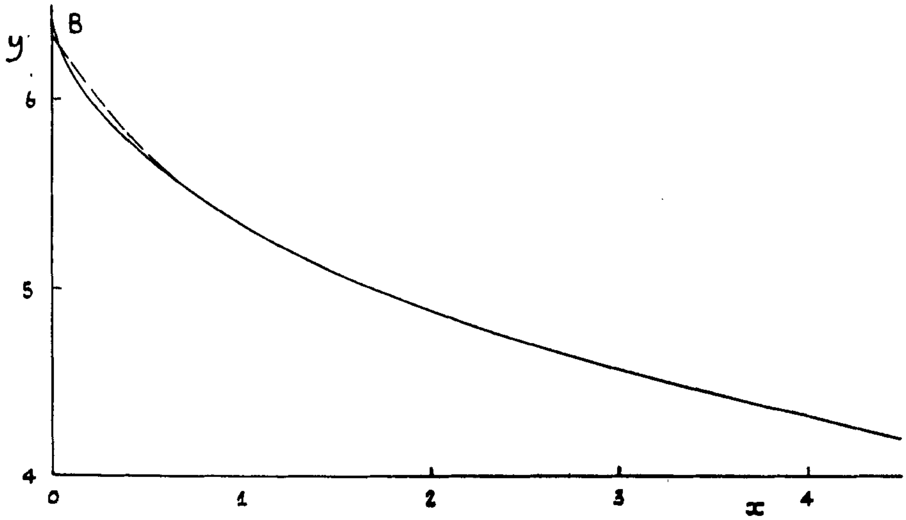

Figure 3. Comparison between free surface profiles given by Southwell & Vaisey (- - -) and by present method (———); $H = 12$ in the units of length used here.

region. The form of Southwell & Vaisey's curve very close to B seems unlikely to be correct, since it requires the streamline to undergo an unduly sudden change of direction at B. It should be noted, however, that figure 3 reproduces only a small part of their complete flow diagram. Even with the fine computational net which they employed in the vicinity of B, errors of the same order of magnitude as the discrepancy noted here are to be expected (the differences between their recorded values of $(H-y)$ and $q/\sqrt{(2g)}$ are of this order). The present method thus appears to be superior (in this one respect) to the relaxation method, even when the latter is applied with the high degree of precision achieved by Southwell & Vaisey.

4. Flow under a planing surface (hydroplane)

We now turn to the problem indicated in figure 4, where the obstacle is a plane inclined at an angle $\beta(<90°)$ to the horizontal. The plane is assumed to have been lowered into the stream to an extent sufficient to avoid standing waves on the downstream side. This problem has received much attention in the past due to its bearing on hydroplanes and sea-plane floats. The scope of existing theories was discussed by Southwell & Vaisey (*loc. cit.*), who also treated an example by relaxation methods. They took $\beta = 30°$, and assumed the same initial conditions as in their sluice-gate example. Their solution shows that the free surface curves very sharply just below the trailing edge B, but beyond this region follows the curve obtained in the previous example (figure 3).

To account for the region near B, we may use the solution of the problem indicated in figure 2(c). This was included in the series of jet problems solved by Mises (1917); but, as in the example previously mentioned, he did not give an expression for the free surface. It can be shown without difficulty, however, that the intrinsic equation of the surface is

$$l = \frac{2d}{\pi} \operatorname{logsec}\left(\frac{\theta+\beta}{2\beta/\pi}\right),$$

where l denotes the length of the arc. Hence, for $\beta = 30°$, we find in the usual way that

$$\frac{y}{d} = 1+\frac{6}{\pi}\left[\frac{1}{\sqrt{3}}\tanh^{-1}\left(\frac{2\sin\theta}{\sqrt{3}}\right)-\sin\theta\right], \tag{4.1}$$

$$= 1{\cdot}229 \quad \text{for} \quad \theta = -30°.$$

The curves obtained from (2.3) and (4.1) are suitably linked at a point where $\theta = -20°$. In the example treated by Southwell & Vaisey, this procedure leads to the value 0·777 for the contraction ratio. The same value is found from the coordinates given in their figure 26. The value in the absence of gravity is, from (4.1), 1/1·229 = 0·814. For a refinement of the method, the solution of the problem in figure 2(d) may be used instead of (4.1).

5. Experimental tests

While in no respect being intended as a complete investigation, the following experiments serve to illustrate some of the matters considered in the preceding sections of this paper. The experiments were carried out in a horizontal channel previously used in the work of Binnie, Davies & Orkney (1955), whose paper can be consulted for full details of the equipment. After the completion of these authors' experiments, the wooden contraction connecting the channel with the supply tank was modified, with the result that uniform subcritical flow could be obtained at higher Froude numbers than before. The channel was 14 in. wide and 8 ft. long, with glass walls about 12 in. high. When necessary, the stream could be held up by means of an adjustable sharp-crested weir at the downstream end; but otherwise it fell freely into a sump after leaving the channel. A movable gantry supported a point gauge and Pitot tube, which provided accurate measurements of the stream depth and total head. Froude numbers were calculated from the formula $F^2 = 2(\text{total head/depth} - 1)$. The water supply was controlled in such a way that the flow rate Q was independent of any obstruction in the channel. This feature assumes significance when the theory of §2 is brought to bear on the experimental observations. The description of the experiments is suitably divided into three parts, as follows,

(i) *Experiments in which the obstacle was fixed*

A rectangular brass plate was contructed with a width slightly less than that of the channel. It was suspended from a horizontal axle mounted above the channel at about 1 ft. from the upstream end, so that it could be swung down broadside to the stream. A sharp edge was machined on the bottom of the plate, and the small gaps between the sides and the walls of the channel were made water-tight with rubber sleeving. Struts attached to the rear face of the plate held it in position at any desired inclination.

In the first experiment, the position of the plate was gradually adjusted until the bottom edge was brought into contact with the surface of a uniform subcritical stream. A large disturbance was always observed as soon as contact was made, provided the Froude number of the undisturbed stream was not too small. An appreciable increase in depth occurred on the upstream side, and a zone of eddying water was formed immediately in front on the obstacle. The latter effect evidently prevented the formation of a stagnation point in the manner indicated by figure 4. When steady conditions were resumed, the obstacle was well immersed in the stream

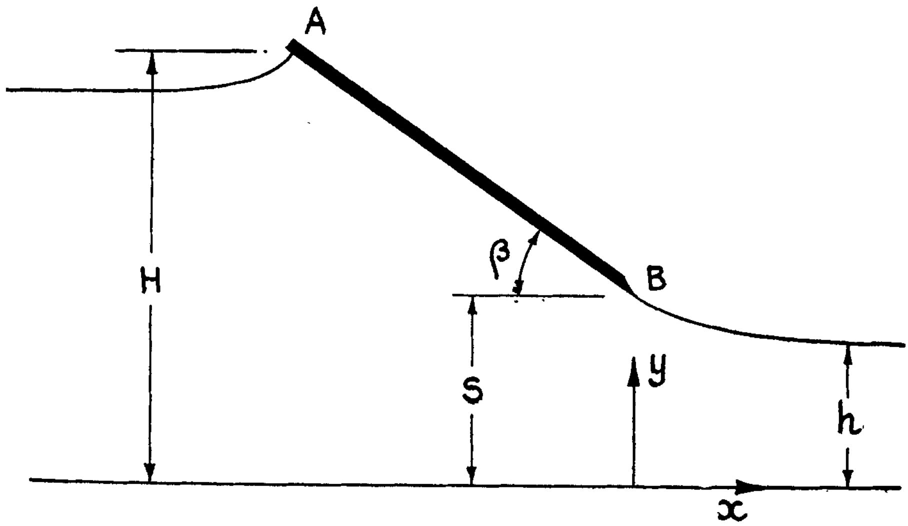

Figure 4. Side elevation of planing surface.

The result on the downstream side varied in character according to the initial Froude number and the inclination of the plate. If the Froude number was fairly small, say less than 0·3, and the plate sloped backwards as in figure 4, a stationary wave-train appeared, its surface being everywhere smooth. At rather higher Froude numbers the waves were larger both in amplitude and length, and the leading wave broke. At a Froude number about 0·7 and with the plate inclined slightly forward, a wave-train appeared immediately after contact but was then swept away, leaving the stream in a uniform supercritical state. The wave-train could be held in place, however, by slightly raising the weir at the channel exit.

The effects of lowering the plate further into the stream are most easily described in reverse, that is, we recount the results of gradually lifting the plate backwards from a vertical position as in figure 1. The adjustments to the plate were made slowly enough for the flow to approximate the steady state appropriate to each stage, and the weir was kept out of action. As the plate was raised, the water level fell upstream and rose downstream, so that the respective Froude numbers increased and decreased. Eventually a wave appeared near the end of the channel and moved upstream, thus causing other waves to form behind it. The leading wave then came to rest just behind the obstacle, and broke at its crest. This exemplified the type of weak undular bore (hydraulic jump) discussed theoretically by Benjamin & Lighthill (1954). By comparing the results of several experiments like this one, the Froude number at which the bore first occurred was found to vary with the depth of the stream. The critical value was about 1·5 for a depth of about 2·5 in., and increased for shallower streams. When the weir was not in use, the bore clearly resulted from the action of friction alone. By raising the weir it could, of course, be made to form at much higher Froude numbers. If the plate was lifted a little further, the turbulent zone at the crest of the leading wave became smaller, and in a few of the tests it vanished entirely. At this stage, however, some sort of instability evidently occurred, for the water level upstream began to rise and fall periodically, while the wave-train downstream began to oscillate longitudinally. The oscillations usually increased in amplitude until the surface of the water suddenly fell away from the bottom of the plate, whereupon the flow became uniform.

According to the theory of § 2, a uniform supercritical stream cannot be formed by the action of an obstacle unless $F > 1\cdot25$, since a wave-train is bound to result from any attempt to reduce F below this value. The experiments indicated that, owing to the formation of bores which is not considered in the theory, the value of of F at which waves first appear is in practice somewhat higher than the theoretical limit. Nevertheless, when waves appear for $F > 1\cdot25$, a significant amount of energy must be dissipated in the bore; whereas in the range $1 < F < 1\cdot25$, a minute amount of dissipation, however small, is sufficient to precipitate a wave-train (see Benjamin & Lighthill, p. 452). It seems likely, therefore, that if measures were taken to reduce frictional effects and so inhibit the formation of a bore (by boundary layer suction, say), the critical value of F could be lowered, but could never be reduced below 1·25.

(ii) *Experiments in which the wave resistance was specified*

To obviate the confusing oscillations observed in the foregoing experiments, the following device was employed. As before, a planing plate was supported from a horizontal axle broadside to the stream. This time, however, the plate was delicately counter-balanced, and a small clearance was left open between each of its sides and the adjacent channel wall. The whole apparatus was of light construction, and the friction of the support was extremely small; consequently a light finger-touch was sufficient to bring the bottom of the plate into contact with the stream. The plate could be immersed to any desired extent by loading the plate with suitable weights; hence it was a simple matter to estimate the horizontal force exerted against the stream. Disturbances arising from the sides of the plate tended to spoil the two-dimensional character of the flow, but were small enough to be unimportant.

By allowing the obstacle freedom of movement in this way, the resistance which it experienced on immersion was made independent of small changes in the level of the stream. This property was, of course, not possessed by the fixed obstacle in the previous experiments, and its absence then was evidently responsible for the instability accompanying slight immersions. No oscillation was observed with the second device in use.

A number of experiments were made in which the resistance was increased gradually from zero by the addition of weights to the plate. The waves first to appear were very small in amplitude, and their length λ was found to be in good agreement with the usual formula

$$F_0^2 = \frac{\lambda}{2\pi h_0} \tanh \frac{2\pi h_0}{\lambda}, \tag{5.1}$$

where h_0 and F_0 are the depth and Froude number upstream. The amplitude of the waves and the wavelength both became larger as the resistance was increased, although the proportional increase in the wavelength was quite small. Eventually the leading wave broke.

The reduction in the quantity S just sufficient to produce breaking waves was estimated for various values of F_0. These estimates were found to agree fairly well with the theoretical results of De (1955), the method of comparison being as follows. If the total drag on the obstacle is M grams weight, the reduction in S is obviously $S^* = Mg/(\rho b)$, where b is the breadth of the channel. In presenting his numerical results, De considers a dimensionless quantity $\mathbf{s} = S/S_c$, where S_c is the value of S for a critical stream with the given flow rate Q. It can easily be shown that

$$S_c = \tfrac{3}{2} g h_0^2 F_0^{4/3}. \tag{5.2}$$

Hence we have, for the reduction in $\mathbf{s}$,

$$\mathbf{s}^* = \frac{2MF_0^{-4/3}}{3\rho h_0^2 b}. \tag{5.3}$$

After specifying F_0, the required theoretical value of S^* can be measured on figure 2 in De's paper. Advantage was taken, however, of an enlarged and more detailed version of this figure kindly made available to the author. A typical set of measurements, made when the leading wave was on the point of breaking, was $h_0 = 14{\cdot}7$ cm, $F_0 = 0{\cdot}701$ and $M = 52$ gm. In addition we had $b = 35{\cdot}6$ cm and $\rho = 1\ \text{gm/cm}^3$, and with these values (5.3) gives $\mathbf{s}^* = 0{\cdot}0072$. The value of $\mathbf{s}^*$ estimated from De's chart was $0{\cdot}0079$. The slight deficiency of the experimental estimate, which is a common feature in all similar tests, may have been due to the well-known fact that in practice stationary waves break before their height reaches the theoretical limit. For the conditions quoted here the wavelength of breaking waves was estimated to be roughly 50 cm. The wave-number $2\pi h_0/\lambda$ was therefore about 1·8, which agrees fairly closely with the value indicated on the chart.

(iii) *Measurements of the contraction ratio for a sluice-gate*

These measurements were taken in order to check the theoretical values derived in § 3. The brass plate used in the first group of experiments was fixed in a vertical position, and careful adjustments were made to insure that the bottom edge was strictly horizontal. The opening s was 3·57 in. for one series of tests, and 1·02 in. for another. In each series, readings of the total head and depth were taken in mid-stream at a distance about 4·5s downstream from the sluice. The value of H was varied over as wide a range as possible by adjusting the flow rate. The maximum useful value of H was that at which the water level in front of the sluice rose to the top of the channel walls. At the minimum value a bore formed on the downstream side.

Experimental values of the contraction ratio h/s are plotted as a function of s/H in figure 5, which includes the theoretical curve according to § 3.

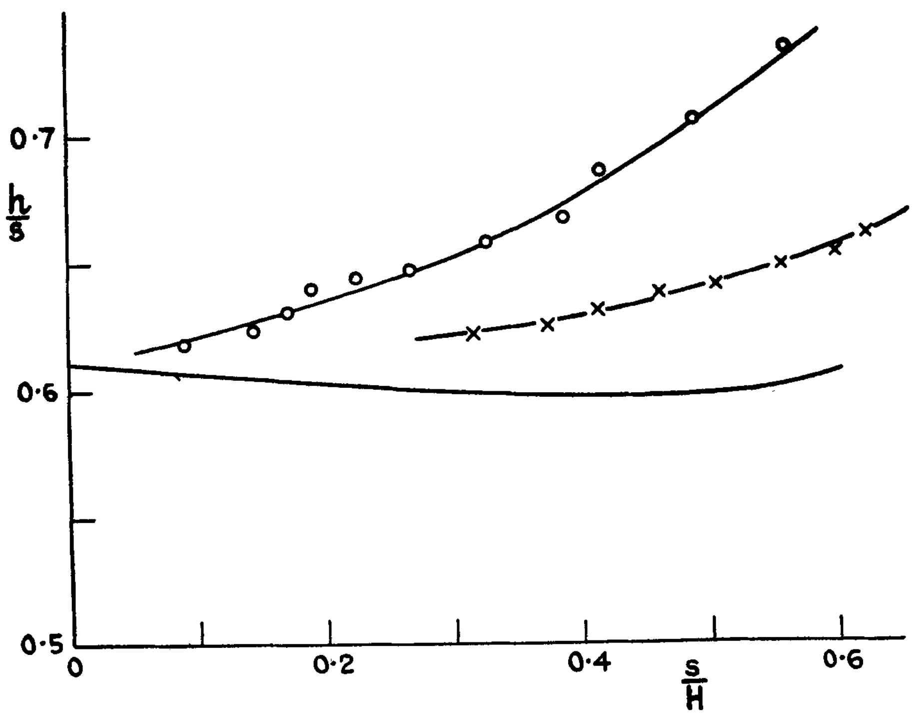

Figure 5. Comparison between theoretical values of contraction ratio (————) and experimental values for sluice-openings of 1·02 in. (——o——) and 3·57 in. (——×——).

Both series of tests specified above are represented in the figure. The two experimental curves are seen to lie above the theoretical curve, the discrepancy being larger for the smaller value of s. The three curves appear to converge for small values of s/H, that is, when the downstream Froude number is large. At first sight figure 5 tends to exaggerate the influence of friction on the flow; but the following interpetation casts a more favourable light on the comparison between theory and experiment.

Following the usual procedure in hydraulic experiments, traces of colouring matter were placed in the stream and their movements observed. In this way the flow-pattern was seen to be approximately as the theory predicts, except in an eddying region near the stagnation point and in the boundary layer on the channel bottom—which was apparently always in a laminar state in the vicinity of the sluice. The boundary layer on the face of the sluice appeared to have only local importance, probably because the velocity is small over most of this region. Let us therefore consider that the discharge under the sluice is affected by friction only in as far as the effective opening is reduced to a value $s-\delta$, where δ is a quantity roughly the same as the thickness of the boundary layer. The depth downstream is similarly reduced; hence the discrepancy Δ in the contraction ratio is approximately proportional to δ/s. If we now tentatively assume that s determines the length scale to be associated with the growth of the boundary layer, it follows that Δ is proportional to the Reynolds number based on the length s. Consequently, for two experiments at the same Froude number, we have $\Delta_1/\Delta_2 = (s_2/s_1)^{3/4}$. With $s_1 = 1{\cdot}02$ and $s_2 = 3{\cdot}57$, the values in inches for the tests in question, this ratio is 2·6. The discrepancy in s/H varies in the same way. Examination of figure 5 shows that the experimental values are roughly in agreement with this result over most of the range of s/H. The extent of the agreement is in fact quite surprising in view of the very rough and ready nature of the above calculation. By observing the drop in total head as the Pitot tube was brought near the channel bottom, the thickness of the boundary layer near the wider sluice ($s = 3{\cdot}57$ in.) was estimated to be of the order of $\frac{1}{4}$ in. Note that the corrections necessary to bring the experimental results into agreement with the theory are also of this order.

These three groups of experiments amply confirm the theory's usefulness in qualitatively assessing the flow. They suggest moreover that if allowance could be made for the boundary layer on the channel bottom, the theory would give accurate predictions of the drag on an obstacle and other physical quantities. The eddying zone in front of the obstacle was the most noticeable effect of friction in the experiments, but it appeared to have little influence on the flow downstream. Very complex phenomena, due to surface tension and shedding of the boundary layer, were to be expected in the region where the free surface first springs clear of the obstacle; but they did not appear to affect the overall characteristics of the flow.

Further experimental work is desirable, especially on the boundary layer. For instance, each of a series of measurements as in (iii) might usefully be accompanied by careful estimates of the boundary layer thickness. This procedure might establish a close agreement with the theory of §2. Further, the method used in (ii) could probably be refined a good deal, and might profitably be extended to measurements of wave resistance over the entire range of De's theoretical results.

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